Math 247A Lecture 15 Notes

Daniel Raban

February 10, 2020

1 Riesz Tranforms and Calderón-Zygmund Convolution Kernels

1.1 Riesz tranforms

Last time, we proved the Sobolev embedding theorem:

Theorem 1.1 (Sobolev embedding)**.** *For $f \in \mathcal{S}(\mathbb{R}^d)$ and $0 < s < d$, we have*

$$\|f\|_q \lesssim \||\nabla|^s f\|_p$$

whenever $\frac{1}{p} = \frac{1}{q} + \frac{s}{d}$. The implicit constant is independent of f.

In particular,

$$\|f\|_q \lesssim \||\nabla| f\|_p, \qquad \frac{1}{p} = \frac{1}{q} + \frac{1}{d}.$$

However, the Fourier transform is not a local operator; it looks at the whole function. However, we can ask whether it is true that

$$\|f\|_q \lesssim \|\nabla f\|_p \qquad \frac{1}{p} = \frac{1}{q} + \frac{1}{d}$$

with $1 < p < q < \infty$. This would follow from boundedness of the Riesz transforms on L^p for $1 < p < \infty$.

Definition 1.1. For $1 \leq j \leq d$ and $f \in \mathcal{S}(\mathbb{R}^d)$, we define the **Riesz transforms** as

$$\widehat{R_j f}(\xi) = m_j(\xi) \widehat{f}(\xi) = -\frac{i\xi_j}{|\xi|} \widehat{f}(\xi).$$

In other words, $R_j = -\frac{\partial_j}{|\nabla|}$.

We write

$$2\pi|\xi| = \sum_{j=1}^{d} m_j(\xi) \cdot 2\pi i \xi_j.$$

So

$$|\nabla| = \sum_{j=1}^{d} R_j \partial_j.$$

Then

$$\|f\|_q \lesssim \||\nabla|\|_p \leq \sum_{j=1}^{d} \|R_j \partial_j f\| \lesssim \sum_{j=1}^{d} \|\partial_j f\|_p \lesssim \|\nabla f\|_p,$$

if we knew the Riesz transforms were bounded on L^p. (The last step comes from the fact that all finite dimensional vector space norms are equivalent.)

Remark 1.1. If we knew that the Riesz transforms are bounded on L^p for $1 < p < \infty$, we could also conclude that the solution u to the Poisson equation $-\Delta u = f$ satisfies $\partial_j \partial_k u \in L^p$ whenever $f \in L^p$. Indeed,

$$(\partial_j \partial_k u)^\wedge(\xi) = -4\pi^2 \xi_j \xi_k \widehat{u}(\xi) = -\frac{\xi_j \xi_k}{|\xi|^2} \widehat{f}(\xi) = m_j(\xi) m_k(\xi) \widehat{f}(\xi).$$

So $\partial_j \partial_k u = R_j R_k f$.

How do we prove boundedness of Riesz transforms?

Definition 1.2. A function $K : \mathbb{R}^d \setminus \{0\} \to \mathbb{C}$ is a **Calderón-Zygmund convolution kernel** if it satisfies:

1. $|K(x)| \lesssim |x|^{-d}$ uniformly for $|x| > 0$.
2. $\int_{R_1 \leq |x| \leq R_2} L(x)\, dx = 0$ for all $0 < R_1 < R_2 < \infty$ (cancellation condition).
3. $\int_{|x| > 2|y|} |K(x+y) - K(x)|\, dx \lesssim 1$ uniformly for $y \in \mathbb{R}^d$ (regularity condition).

Example 1.1. The Riesz tranforms correspond to Calderón-Zygmund convolution kernels.

$$m_j = -\frac{i\xi_j}{|\xi|} \implies k_j(x) = m_j^\vee(x) = -\frac{1}{2\pi} \partial_j \left[\frac{\pi^{-(d-1)/2} \Gamma((d-1)/2)}{\pi^{-1/2} \Gamma(1/2)} \right]^{(d-1/2)} \sim_d \frac{x_j}{|x|^{d+1}}.$$

We have

1. $|k_j(x)| \lesssim x^{-d}$ uniformly in $|x| > 0$.

2. $\int_{R_1\le|x|\le R_2} k_j(x)\,dx = 0$ for all $0<R_1<R_2<\infty$ because it is odd in x_j.

3. By the fundamental theorem of calculus,

$$\int_{|x|\ge 2|y|} |k_j(x+y)-k_j(x)|\,dx \le \int_{|x|>2|y|} |y| \int_0^1 |\nabla k_j(x+\theta y)|\,d\theta\,dx$$

For $|x|>2y|$ and $\theta\in(0,1)$, $|x|/2 \le |x|-|y| \le |x+\theta y| \le |x|+|y| \le 3|x|/2$.

$$\begin{aligned}&\lesssim \int_{|x|>2|y|} |y| \frac{1}{|x|^{d+1}}\,dx\\ &\lesssim |y|\cdot\frac{1}{|y|} \lesssim 1.\end{aligned}$$

More generally, we have proved the following.

Lemma 1.1. *If $|\nabla K(x)| \lesssim |x|^{-(d+1)}$ uniformly for $|x|>0$, then K satisfies the regularity condition.*

1.2 L^2-boundedness of convolution with Calderón-Zygmund convolution kernels

Here is a lemma we need.

Lemma 1.2. *Let $K:\mathbb{R}^d\setminus\{0\}\to\mathbb{C}$ be a Calderón-Zygmund convolution kernel. For $\varepsilon>0$, let $K_\varepsilon = K\mathbb{1}_{\{\varepsilon\le|x|\le 1/\varepsilon\}}$. Then K_ε is a Calderón-Zygmund convolution kernel.*

Proof. $|k_\varepsilon(x)| \lesssim |k(x)| \lesssim |x|^{-d}$ uniformly for $|x|>0$. For the second condition,

$$\int_{R_1\le|x|\le R_2} K_\varepsilon(x)\,dx = \int_{\max\{R_1,\varepsilon\}\le|x|\le\min\{R_2,1/\varepsilon\}} K(x)\,dx = 0, \qquad \forall 0<R_1<R_2<\infty.$$

For the third condition,

$$\begin{aligned}\int_{|x|>2|y|} |K_\varepsilon(x+y)-K_\varepsilon(y)|\,dx \le &\int_{\substack{\varepsilon\le|x|\le 1/\varepsilon\\ \varepsilon\le|x+y|\le 1/\varepsilon\\ |x|>2|y|}} |K_\varepsilon(x+y)-K_\varepsilon(y)|\,dx\\ &+\int_{\substack{\varepsilon\le|x|\le 1/\varepsilon\\ \varepsilon>|x+y| \text{ or } |x+y|>1/\varepsilon\\ |x|>2|y|}} |K_\varepsilon(x)|\,dx\\ &+\int_{\substack{\varepsilon>|x| \text{ or } x>1/\varepsilon\\ \varepsilon\le|x+y|\le 1/\varepsilon\\ |x|>2|y|}} |K_\varepsilon(x+y)|\,dx.\end{aligned}$$

Look at I: If $|x+y| < \varepsilon$, then $|x| \le |x+y| + |y| \le |x+y| + |x|/2$, so $|x| \le 2|x+y| \le 2\varepsilon$. The contribution is at most

$$\int_{\varepsilon \le |x| \le 2\varepsilon} |K(x)|\, dx \lesssim \int_{\varepsilon \le |x| \le 2\varepsilon} |x|^{-d}\, dx \lesssim 1,$$

uniformly in $\varepsilon > 0$.

If $|x+y| > 1/\varepsilon$, then $|x| \ge |x+y| - |y| \ge |x+y| - |x|/2$, so $|x| \ge \frac{2}{3}|x+y| \ge \frac{2}{3\varepsilon}$. The contribution is at most

$$\int_{\frac{2}{3\varepsilon} \le |x| \le \frac{1}{\varepsilon}} |K(X)|\, dx \lesssim 1,$$

uniformly in $\varepsilon > 0$. Similarly, II $\lesssim 1$ uniformly in $\varepsilon > 0$ and $y \in \mathbb{R}^d$. □

Theorem 1.2. *Ket $K : \mathbb{R}^d \setminus \{0\} \to \mathbb{C}$ be a Calderón-Zygmund convolution kernel. For $\varepsilon > 0$, let $K_\varepsilon = K\mathbb{1}_{\{\varepsilon < |x| \le 1/\varepsilon\}}$. THen*

$$\|K_\varepsilon * f\|_2 \lesssim \|f\|_2$$

uniformly for $\varepsilon > 0, f \in L^2$. Consequently, $f \mapsto K * f$ (which is the L^2 limit as $\varepsilon \to 0$ of $K_\varepsilon * f$) extends continuously from $\mathcal{S}(\mathbb{R}^d)$ to a bounded map on $L^2(\mathbb{R}^d)$.

Proof.

$$\begin{aligned}
\|K_\varepsilon * f\|_2 &= \|\widehat{K_\varepsilon * f}\|_2 \\
&= \|\widehat{K}_\varepsilon \widehat{f}\|_2 \\
&\le \|\widehat{K}_\varepsilon\|_\infty \|whf\|_2 \\
&\le \|\widehat{K}_\varepsilon\|_\infty \|f\|_2.
\end{aligned}$$

Fix $\xi \in \mathbb{R}^d$. Then

$$\begin{aligned}
\widehat{K}_\varepsilon(\xi) &= \int e^{-2\pi i x\cdot\xi} K_\varepsilon(x)\, dx \\
&= \int_{|x| \le 1/|\xi|} e^{-2\pi i x\cdot\xi} K_\varepsilon(x)\, dx + \int_{|x| > 1/|\xi|} e^{-2\pi i x\cdot\xi} K_\varepsilon(x)\, dx.
\end{aligned}$$

Now observe that by condition 2 of the definition of the Calderón-Zygmund convolution kernel,

$$\left| \int_{\varepsilon \le |x| \le 1/|\xi|} e^{-2\pi i x\cdot\xi} K_\varepsilon(x)\, dx \right| = \left| \int_{|x| \le 1/|\xi|} (e^{-2\pi i x\cdot\xi} - 1) K_\varepsilon(x)\, dx \right|$$

By condition 1,

$$\lesssim \int_{|x| \le 1/|\xi|} |x||\xi||x|^{-d}\, dx$$

$$
\begin{aligned}
&\lesssim |\xi| \frac{1}{|\xi|} \\
&\lesssim 1.
\end{aligned}
$$

On the other hand, we have

$$
\begin{aligned}
\int_{|x|>1/|\xi|} e^{-2\pi i x\cdot\xi} K_\varepsilon(x)\, dx &= \int_{|x|>1/|\xi|} \frac{1}{2}(1-e^{\pi i})e^{-2\ piix\cdot\xi} K_\varepsilon(x)\, dx \\
&= \int_{|x|>1/|\xi|} \frac{1}{2} e^{-2\pi i x\cdot\xi} K_\varepsilon(x)\, dx \\
&\quad - \frac{1}{2}\int_{|x|>1/|\xi|} e^{-2\pi i\xi(x-\xi/(2|\xi|^2)} K_\varepsilon(x)\, dx \\
&= \int_{|x|>1/|\xi|} \frac{1}{2} e^{-2\pi i x\cdot\xi} K_\varepsilon(x)\, dx \\
&\quad - \frac{1}{2}\int_{\left|x+\frac{\xi}{2|\xi|^2}\right|>\frac{1}{|\xi|}} e^{-2\pi i x\cdot\xi} K_\varepsilon\left(x+\frac{|xi|}{2|\xi|^2}\right)\, dx,
\end{aligned}
$$

which puts us into a position to make a change of variables and use condition 3. We will finish the proof next time. □